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# Stability of the high-order exponential finite difference method for two-dimensional nonlinear elliptic problems via discrete maximum principle

Estabilidade do método de diferenças finitas exponencial de alta ordem para problemas elípticos não lineares bidimensionais via princípio do máximo discreto

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## ABSTRACT

This work analyzes the consistency, stability, and convergence of the high-order exponential finite difference method (EFDM) applied to two-dimensional nonlinear elliptic problems under Dirichlet boundary conditions on a domain $\Omega = (a, b) \times (a, b)$. The approach employs a nine-point stencil on the discretization grid to approximate the Laplacian operator, increasing the order of accuracy relative to conventional five-point schemes. Through a Taylor series expansion truncated up to the sixth order, it is shown that the local truncation error is of fourth order, ensuring the consistency of the discretization. Additionally, the discrete maximum principle is established for the proposed problem, and an *a priori* estimate for nine-point operators is derived, which rigorously guarantees the scheme's stability and provides a bound for the global error. It is concluded that the high-order exponential method is consistent, stable, and highly efficient for solving the analyzed class of problems.

**Keywords:** Nonlinear elliptic equations; exponential finite difference method; discrete maximum principle; high order of convergence; interdisciplinarity.

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**RESUMO**

Este trabalho analisa a consistência, estabilidade e convergência do método de diferenças finitas exponencial de alta ordem (MDFE) aplicado a problemas elípticos não lineares bidimensionais sob condições de fronteira de Dirichlet em um domínio $\Omega = (a, b) \times (a, b)$. A abordagem emprega um estêncil de nove pontos na malha de discretização para aproximar o operador Laplaciano, elevando a ordem de precisão em relação aos esquemas convencionais de cinco pontos. Mediante o desenvolvimento em série de Taylor com truncamento até a sexta ordem, comprova-se que o erro de truncamento local é de quarta ordem, assegurando a consistência da discretização. Adicionalmente, estabelece-se o princípio do máximo discreto para o problema proposto e deriva-se uma estimativa *a priori* para operadores de nove pontos, o que garante rigorosamente a estabilidade do esquema e fornece um limite para o erro global. Conclui-se que o método exponencial de alta ordem é consistente, estável e altamente eficiente para a resolução da classe dos problemas analisados.

**Palavras-chave:** Equações elípticas não lineares; método de diferenças finitas exponencial; princípio do máximo discreto; alta ordem de convergência; interdisciplinaridade.

# 1 INTRODUCTION

Elliptic partial differential equations (PDEs) play a fundamental role in modeling various physical, engineering, and applied science phenomena, such as steady-state heat conduction problems, electrostatics, fluid mechanics, and elasticity. In many real-world situations, these equations exhibit nonlinear characteristics over two-dimensional domains, making it impossible to obtain analytical solutions in terms of elementary functions. Consequently, the development and analysis of efficient, stable, and high-precision numerical methods become indispensable for approximating the solutions to these problems (Thomas, 1995).

Among the widely used discretization techniques, the finite difference method (FDM) (Cuminato and Junior, 2013) stands out due to its conceptual simplicity and ease of implementation. However, conventional finite difference schemes – such as the standard formulation based on a five-point stencil – exhibit an order of convergence limited to $O(h^2)$,

where $h$ represents the mesh discretization step size. To achieve higher accuracy without requiring excessive mesh refinement (which would significantly increase computational cost), high-order schemes based on nine-point stencils have been developed and explored in the literature.

In this context, the EFDM approach proposed by (Pandey, 2013) stands out. This technique combines the use of compact nine-point stencils with exponential terms derived from meticulous Taylor series expansions. As a result, a significant gain in the accuracy of the Laplacian operator approximation is achieved, raising the convergence rate to $O(h^4)$.

The main objective of this work is to formally investigate the stability, consistency, and convergence of the EFDM applied to a two-dimensional nonlinear elliptic equation subject to Dirichlet boundary conditions on a bounded domain $\Omega = (a, b) \times (a, b)$. The modeled problem is given by:

$$\Delta u(x, y) = -f(x, y, u(x, y)), \quad (x, y) \in \Omega, \tag{1.1}$$

subject to the boundary condition:

$$u(x, y) = g(x, y), \ (x, y) \in \partial\Omega, \tag{1.2}$$

where $\partial\Omega$ denotes the boundary of the domain, $f$ and $g$ are continuous real-valued functions (with $f$ not identically zero), and $\Delta$ is the two-dimensional Laplacian operator:

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y). \tag{1.3}$$

The existence and uniqueness of a solution for this class of non-linear elliptic equations, under suitable regularity conditions on $f$ and $g$, are consolidated results in the classical theory of PDEs, see for example (Evans, 2022 and, Gilbarg and Trudinger, 1977). For the analysis and theoretical guarantee of the stability of the numerical solution obtained by the nine-point operator, use is made of the Discrete Maximum Principle and the derivation of an *a priori* estimate. From these theoretical results, it is demonstrated that the global error of the method is of the order of $O(h^4)$, rigorously proving the convergence and efficiency of the studied numerical scheme.

## 2 PROBLEM FORMULATION

In order to find a numerical solution to equation (1.1), with boundary conditions (1.2), suppose that $\Omega$ is covered by a mesh, with each arbitrary point of this mesh given by $(x_i, y_j)$, in which

$$x_i = a + ih \text{ and } y_j = a + jh,$$

with $i, j = 0, 1, ..., N + 1$ and $h = \frac{b-a}{N+1}$.

We denote by $\Omega_\delta$ the set of mesh points interior to $\Omega$ and by $\partial\Omega_\delta$ the set of mesh points belonging to $\partial\Omega$.

In order to increase the order of convergence relative to the conventional five-point scheme, this section presents the EFDM, based on a compact nine-point stencil. This approach to the two-dimensional Dirichlet problem is inspired by the works of (Cuminato and Junior, 2013) and (Pandey, 2013).

For $i, j = 0, 1, ..., N + 1$, we consider nine mesh points given by $(i, j)$, $(i\pm1, j)$, $(i, j\pm1)$ and $(i\pm1, j\pm1)$, and we denote:

$$u_{i,j} = u(x_i, y_j); \quad u_{i+1,j} = u(x_i + h, y_j); \quad u_{i-1,j} = u(x_i - h, y_j);$$

$$u_{i,j+1} = u(x_i, y_j + h); \quad u_{i,j-1} = u(x_i, y_j - h); \quad u_{i+1,j+1} = u(x_i + h, y_j + h);$$

$$u_{i-1,j+1} = u(x_i - h, y_j + h); \; u_{i+1,j-1} = u(x_i + h, y_j - h); \; u_{i-1,j-1} = u(x_i - h, y_j - h).$$

As in (Pandey, 2013), considering $f_{i,j} = -f(x_i, y_j, u_{i,j})$, the EFDM is given by

$$4(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1})$$

$$+ (U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i-1,j-1}) - 20U_{i,j} = 6h^2 f_{i,j} e^{\frac{h^2\Delta f_{i,j}}{12f_{i,j}}}, \quad (2.1)$$

or alternatively,

$$\Delta_\delta U_{i,j} = f_{i,j} e^{\frac{h^2\Delta f_{i,j}}{12f_{i,j}}} \; in \; \Omega_\delta, \quad (2.2)$$

where $\Delta_\delta$ is the nine-point discrete Laplacian operator defined as

$$\Delta_\delta U_{i,j} = \frac{1}{6h^2}\left[4\left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right)\right.$$
$$\left. + \left(U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i-1,j-1}\right) - 20\,U_{i,j}\right] \qquad (2.3)$$

with $i,\ j = 1, \ldots, N$.

The Laplacian $\Delta f_{i,j}$ is given by the five-point difference formula, that is,

$$\Delta f_{i,j} = \frac{f_{i+1,j}+f_{i-1,j}-4f_{i,j}+f_{i,j+1}+f_{i,j-1}}{h^2}.$$

Thus,

$$h^2\Delta f_{i,j} = f_{i+1,j} + f_{i-1,j} - 4f_{i,j} + f_{i,j+1} + f_{i,j-1}.$$

For a discrete function $V: \Omega_\delta \cup \partial\Omega_\delta \to R$, consider the operator $\Delta_\delta V: \Omega_\delta \to R$ defined by

$$\Delta_\delta V(x, y) = \frac{1}{6h^2}\left[4\left(V(x+h, y) + V(x-h, y) + V(x, y+h) + V(x, y-h)\right)\right.$$
$$- 20\,V(x, y) + V(x+h, y+h) + V(x-h, y+h)$$
$$\left. + V(x+h, y-h) + V(x-h, y-h)\right]. \qquad (2.4)$$

In the following section, the local truncation error analysis of the numerical method (2.1) will be performed, demonstrating that its order of accuracy is $O(h^4)$.

## 3. CONSISTENCY

The local truncation error at the mesh point $(x_i, y_j)$ is defined as

$$T_{i,j} = \Delta_\delta u_{i,j} - f_{i,j}\, e^{\frac{\pi^2 \Delta f_{i,j}}{12 f_{i,j}}}. \tag{3.1}$$

**Lemma 3.1.** *If the solution u is sufficiently differentiable, with partial derivatives*

$$\frac{\partial^6 u}{\partial x^6},\ \frac{\partial^6 u}{\partial x^4 \partial y^2},\ \frac{\partial^6 u}{\partial x^2 \partial y^4} \text{ and } \frac{\partial^6 u}{\partial y^6}$$

*bounded, then the local truncation error* $T_{i,j}$, *defined in* (3.1), *satisfies*

$$\left|T_{i,j}\right| \leq C\, h^4, \tag{3.2}$$

*for some positive constant* $C$, *independent of* $h$.

**Proof.** Using the Taylor series expansion of the function $u$ around the mesh point $(x_i, y_j)$ and truncating this expansion up to the sixth order, we obtain:

$$u_{i+1,j} = u_{i,j} + \sum_{k=1}^{6} \frac{h^k}{k!} \frac{\partial^k u}{\partial x^k}(x_i, y_j) + O(h^7);$$

$$u_{i-1,j} = u_{i,j} + \sum_{k=1}^{6} \frac{(-1)^k h^k}{k!} \frac{\partial^k u}{\partial x^k}(x_i, y_j) + O(h^7);$$

$$u_{i,j+1} = u_{i,j} + \sum_{k=1}^{6} \frac{h^k}{k!} \frac{\partial^k u}{\partial y^k}(x_i, y_j) + O(h^7);$$

$$u_{i,j-1} = u_{i,j} + \sum_{k=1}^{6} \frac{(-1)^k h^k}{k!} \frac{\partial^k u}{\partial y^k}(x_i, y_j) + O(h^7).$$

Adding these four equalities, we obtain

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = 4\,u_{i,j} + h^2\left( \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j)\right)$$

$$+ \frac{h^4}{12}\left( \frac{\partial^4 u}{\partial x^4}(x_i, y_j) + \frac{\partial^4 u}{\partial y^4}(x_i, y_j)\right) + \frac{h^6}{360}\left(\frac{\partial^6 u}{\partial x^6}(x_i, y_j) + \frac{\partial^6 u}{\partial y^6}(x_i, y_j)\right) + O(h^7).$$

Hence, from (1.1) and the fact that $f_{i,j} = \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j)$, we obtain

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = 4u_{i,j} + h^2 f_{i,j}$$
$$+ \frac{h^4}{12}\left(\Delta f_{i,j} - 2\frac{\partial^4 u}{\partial x^2 \partial y^2}(x_i, y_j)\right) + \frac{h^6}{360}\left(\frac{\partial^6 u}{\partial x^6}(x_i, y_j) + \frac{\partial^6 u}{\partial y^6}(x_i, y_j)\right) + O(h^7). \qquad (3.3)$$

Employing once again the Taylor series expansion of the function $u$, around the point $(x_i, y_j)$, and truncating this expansion up to the sixth order, we have:

$$u_{i-1,j+1} = u_{i,j} + \sum_{k=1}^{6} \frac{h^k}{k!} \sum_{r=0}^{k} (-1)^{k-r} (k\,r) \frac{\partial^k u}{\partial x^{k-r} \partial y^r}(x_i, y_j) + O(h^7);$$

$$u_{i+1,j-1} = u_{i,j} + \sum_{k=1}^{6} \frac{h^k}{k!} \sum_{r=0}^{k} (-1)^{k-r} (k\,r) \frac{\partial^k u}{\partial x^r \partial y^{k-r}}(x_i, y_j) + O(h^7);$$

$$u_{i-1,j-1} = u_{i,j} + \sum_{k=1}^{6} \frac{(-h)^k}{k!} \sum_{r=0}^{k} (k\,r) \frac{\partial^k u}{\partial x^r \partial y^{k-r}}(x_i, y_j) + O(h^7).$$

Summing the previous equalities, using (1.1) and the fact that

$$\frac{\partial^4 u}{\partial x^4}(x_i, y_j) + \frac{\partial^4 u}{\partial y^4}(x_i, y_j) = \Delta f_{i,j} - 2\frac{\partial^4 u}{\partial x^2 \partial y^2}(x_i, y_j),$$

we have

$$u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}$$
$$= 4u_{i,j} + 2h^2\left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j)\right]$$
$$+ \frac{h^4}{6}\left[\frac{\partial^4 u}{\partial x^4}(x_i, y_j) + 6\frac{\partial^4 u}{\partial x^2 \partial y^2}(x_i, y_j) + \frac{\partial^4 u}{\partial y^4}(x_i, y_j)\right]$$
$$+ \frac{h^6}{180}\left[\frac{\partial^6 u}{\partial x^6}(x_i, y_j) + 15\frac{\partial^6 u}{\partial x^4 \partial y^2}(x_i, y_j) + 15\frac{\partial^6 u}{\partial x^2 \partial y^4}(x_i, y_j) + \frac{\partial^6 u}{\partial y^6}(x_i, y_j)\right] + O(h^7)$$

$$= 4\,u_{i,j} + 2\,h^2 f_{i,j} + \frac{h^4}{6}\left[\Delta f_{i,j} + 4\frac{\partial^4 u}{\partial x^2\,\partial y^2}\left(x_i, y_j\right)\right]$$

$$+ \frac{h^6}{180}\left[\frac{\partial^6 u}{\partial x^6}\left(x_i, y_j\right) + 15\,\frac{\partial^6 u}{\partial x^4\,\partial y^2}\left(x_i, y_j\right) + 15\,\frac{\partial^6 u}{\partial x^2\,\partial y^4}\left(x_i, y_j\right)\right.$$

$$\left. + \frac{\partial^6 u}{\partial y^6}\left(x_i, y_j\right)\right] + O\left(h^7\right). \tag{3.4}$$

From (3.3) and (3.4), we conclude that

$$4\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) + \left(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}\right)$$

$$- 20\,u_{i,j}$$

$$= 6\,h^2 f_{i,j} + \frac{h^4}{2}\Delta f_{i,j} + \frac{h^6}{60}\left[\frac{\partial^6 u}{\partial x^6}\left(x_i, y_j\right) + 5\,\frac{\partial^6 u}{\partial x^4\,\partial y^2}\left(x_i, y_j\right) + 5\,\frac{\partial^6 u}{\partial x^2\,\partial y^4}\left(x_i, y_j\right)\right.$$

$$\left. + \frac{\partial^6 u}{\partial y^6}\left(x_i, y_j\right)\right] + O\left(h^7\right). \tag{3.5}$$

As a consequence of (3.5), we see that

$$\Delta_\delta u_{i,j} = \frac{1}{6\,h^2}\left[6\,h^2 f_{i,j} + \frac{h^4}{2}\Delta f_{i,j} + \frac{h^6}{60}\left(\frac{\partial^6 u}{\partial x^6}\left(x_i, y_j\right) + 5\,\frac{\partial^6 u}{\partial x^4\,\partial y^2}\left(x_i, y_j\right)\right.\right.$$

$$\left. + 5\,\frac{\partial^6 u}{\partial x^2\,\partial y^4}\left(x_i, y_j\right) + \frac{\partial^6 u}{\partial y^6}\left(x_i, y_j\right)\right) + O\left(h^7\right)\Big]$$

$$= f_{i,j} + \frac{h^2}{12}\Delta f_{i,j} + \frac{h^4}{360}\left(\frac{\partial^6 u}{\partial x^6}\left(x_i, y_j\right) + 5\,\frac{\partial^6 u}{\partial x^4\,\partial y^2}\left(x_i, y_j\right) + 5\,\frac{\partial^6 u}{\partial x^2\,\partial y^4}\left(x_i, y_j\right)\right.$$

$$\left. + \frac{\partial^6 u}{\partial y^6}\left(x_i, y_j\right)\right) + O\left(h^5\right) \tag{3.6}$$

Using (3.1), (3.6) and replacing the term $f_{i,j}\,e^{\frac{h^2\Delta f_{i,j}}{12 f_{i,j}}}$ with the approximation $f_{i,j} + \frac{h^2}{12}\Delta f_{i,j}$ obtained from the first two terms of the Maclaurin series expansion of $e^{\frac{h^2\Delta f_{i,j}}{12 f_{i,j}}}$, we conclude that

$$T_{i,j} = \frac{h^4}{360}\left(\frac{\partial^6 u}{\partial x^6}(x_i, y_j) + 5\frac{\partial^6 u}{\partial x^4 \partial y^2}(x_i, y_j) + 5\frac{\partial^6 u}{\partial x^2 \partial y^4}(x_i, y_j) + \frac{\partial^6 u}{\partial y^6}(x_i, y_j)\right).$$

Using this with the hypotheses on the partial derivatives, we obtain (3.2), which completes the proof. Therefore, the method given in (2.1) is consistent and of order $h^4$.

## 4. STABILITY AND CONVERGENCE

The goal now is to show that method (2.2) is stable. To this end, we will establish some results. Initially, we define the *global error* at the mesh point $(x_i, y_j)$ as

$$E_{i,j} = u_{i,j} - U_{i,j}, \qquad (4.1)$$

where $u_{i,j}$ and $U_{i,j}$ denote the exact solution and the numerical solution at the point $(x_i, y_j)$, respectively.

Next, we will prove the discrete maximum principle (Strikwerda, 2004).

**Theorem 4.1 (Discrete Maximum Principle).**

*(i) If* $V: \Omega_\delta \cup \partial\Omega_\delta \to R$ *is a discrete function and satisfies*

$$\Delta_\delta V(x, y) \geq 0, \ for\ all\ (x, y) \in \Omega_\delta,$$

*then*

$$V(x, y) \leq V(x, y).$$

*(ii) Alternatively, if* $V: \Omega_\delta \cup \partial\Omega_\delta \to R$ *satisfies*

$$\Delta_\delta V(x, y) \leq 0, \ for\ all\ (x, y) \in \Omega_\delta,$$

*then*

$$V(x,\ y) \geq V(x,\ y).$$

**Proof.** In item (*i*), suppose that there exists $P_0 = (x_r,\ y_s) \in \Omega_\delta$ such that $V(P_0) = M_0$, where $V(P) \leq M_0$, $\forall\ P \in \Omega_\delta$ and $V(P) < M_0$, $\forall\ P \in \partial\Omega_\delta$.

Next, consider the points

$$P_1 = (x_r + h,\ y_s), \quad P_2 = (x_r - h,\ y_s), \quad P_3 = (x_r,\ y_s + h), \quad P_4 = (x_r,\ y_s - h),$$

$$P_5 = (x_r + h,\ y_s + h),\ P_6 = (x_r - h,\ y_s + h),\ P_7 = (x_r + h,\ y_s - h) \text{ and } P_8 = (x_r - h,\ y_s - h).$$

Using the hypothesis and definition (2.3) for the operator $\Delta_\delta$, we obtain

$$\Delta_\delta V(P_0) = \frac{1}{6h^2}\,[4\,(V(P_1) + V(P_2) + V(P_3) + V(P_4)) - 20\,V(P_0) + V(P_5) + V(P_6) + V(P_7) + V(P_8)] \geq 0.$$

This implies that

$$20\,M_0 \leq 4\,(V(P_1) + V(P_2) + V(P_3) + V(P_4)) + V(P_5) + V(P_6) + V(P_7) + V(P_8). \qquad (4.2)$$

Since $V(Q) \leq M_0$, for all $Q \in \Omega_\delta \cup \partial\Omega_\delta$, we conclude that

$$V(P_\nu) = M_0,$$

for any $\nu \in \{1,\ \cdots,\ 8\}$. Indeed, suppose that $V(P_\nu) < M_0$ for some $\nu \in \{1,\ \cdots,\ 8\}$. Then, from (4.2), we deduce that

$$20\,M_0 < 4\,(M_0 + M_0 + M_0 + M_0) + M_0 + M_0 + M_0 + M_0 = 20\,M_0.$$

Therefore, $V(P_\nu) = M_0$, for all $\nu \in \{1,\ \cdots,\ 8\}$. Repeating this argument for each point $P_\nu \in \Omega_\delta$ instead of $P_0$, we obtain

$$V(P) = M_0, \text{ for all } P \in \Omega_\delta \cup \partial\Omega_\delta.$$

But this contradicts the assumption that $V(P) < M_0$, for all $P \in \partial\Omega_\delta$. Thus, the proof of item (*i*) is completed.

(*ii*) Note that $(-V) = -V$ and $\Delta_\delta(-V) = -\Delta_\delta V$. Since $\Delta_\delta V(x, y) \leq 0$, for all $(x, y) \in \Omega_\delta$, we have $\Delta_\delta(-V(x, y)) \geq 0$, for all $(x, y) \in \Omega_\delta$. By item (*i*), we obtain

$$[-V(x, y)] \leq [-V(x, y)]$$

and, consequently,

$$V(x, y) \geq V(x, y).$$

Therefore, item (*ii*) is proven, and this completes the proof of the theorem.

**Remark 4.1.** *By Theorem 4.1, if the maximum, in case (i), or the minimum, in case (ii), of V occurs in* $\Omega_\delta$, *then V is constant in* $\Omega_\delta \cup \partial\Omega_\delta$.

Indeed, suppose that there exists $P_0 = (x_r, y_s) \in \Omega_\delta$ such that

$$V(x, y)$$

and consider the points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$ and $P_8$ as in the beginning of the proof of Theorem 4.1. Based on the above assumption, $V(P) \leq M = V(P_0)$, for all $P \in \Omega_\delta \cup \partial\Omega_\delta$. Using the definition of the operator $\Delta_\delta$, we obtain

$$\Delta_\delta V(P_0) = \frac{1}{6h^2}[4(V(P_1) + V(P_2) + V(P_3) + V(P_4)) - 20V(P_0) + V(P_5) + V(P_6) + V(P_7) + V(P_8)] \leq 0,$$

since $V(P_i) \leq V(P_0)$, for all $i \in \{1, \cdots, 8\}$.

Since, by hypothesis, $\Delta_\delta V(P_0) \geq 0$, we conclude that $\Delta_\delta V(P_0) = 0$. This implies that

$$M = \frac{1}{20}[4\left(V(P_1) + V(P_2) + V(P_3) + V(P_4)\right) + V(P_5) + V(P_6) + V(P_7) + V(P_8)].$$

Since $V(P) \leq M$, for all $P \in \Omega_\delta \cup \partial\Omega_\delta$, we see that

$$V(P_\nu) = M, \text{ for all } \nu \in \{1, \cdots, 8\}.$$

Indeed, if there were some $\nu \in \{1, \cdots, 8\}$ for which $V(P_\nu) < M$, we would have $M < M$, which is a contradiction. Thus, $V(P_\nu) = M$ holds. Next, it suffices to repeat this argument for each of the points $P_\nu \in \Omega_\delta$ in place of $P_0$. By repetition, each point of $\Omega_\delta$ and $\partial\Omega_\delta$ appears as one of the points $P_\nu$ for some corresponding $P_0$. Thus, we conclude that

$$V(P) = M, \text{ for all } P \in \Omega_\delta \cup \partial\Omega_\delta. \qquad \blacksquare$$

The next result provides a bound for the solution to equation (2.2).

**Theorem 4.2 (A Priori Estimate).** If $V: \Omega_\delta \cup \partial\Omega_\delta \to R$ is a discrete function, then

$$\max_{(x,y)\in\Omega_\delta} |V(x, y)| \leq \max_{(x,y)\in\partial\Omega_\delta} |V(x, y)| + \frac{a^2+b^2}{2} \max_{(x,y)\in\Omega_\delta} |\Delta_\delta V(x, y)|. \tag{4.3}$$

**Proof.** Consider the discrete function $\phi: \Omega_\delta \cup \partial\Omega_\delta \to R$ defined by $\phi(x, y) = \frac{1}{2}x^2$. Note that,

$$(x, y) \in \Omega_\delta \cup \partial\Omega_\delta \subset [a, b] \times [a, b],$$

$$0 \leq \phi(x, y) \leq \frac{a^2+b^2}{2}, \text{ for all } (x, y) \in \Omega_\delta \cup \partial\Omega_\delta.$$

Furthermore, for all $(x, y) \in \Omega_\delta \cup \partial\Omega_\delta$,

$$\Delta_\delta \phi(x, y) = 1. \tag{4.4}$$

Now, we define the discrete functions $V_+: \Omega_\delta \cup \partial\Omega_\delta \longrightarrow R$ and $V_-: \Omega_\delta \cup \partial\Omega_\delta \longrightarrow R$ by

$$V_+(x,\ y) = V(x,\ y) + N_0\,\phi(x,\ y)$$

and

$$V_-(x,\ y) = -\ V(x,\ y) + N_0\,\phi(x,\ y),$$

where

$$N_0 = \max_{(x,\ y)\in\Omega_\delta} \left|\Delta_\delta V(x,\ y)\right|.$$

As a consequence, for all $(x,\ y) \in \Omega_\delta$,

(*a*) $\Delta_\delta V_+(x,\ y) = \Delta_\delta V(x,\ y) + N_0 \geq 0$

and

(*b*) $\Delta_\delta V_-(x,\ y) = -\ \Delta_\delta V(x,\ y) + N_0 \geq 0.$

**Verification of (*a*):** Using the definition of the operator $\Delta_\delta$ and (4.4), we have

$$\begin{aligned}\Delta_\delta V_+(x,\ y) = \frac{1}{6h^2}\,[&4\,(V(x+h,\ y) + V(x-h,\ y) + V(x,\ y+h) + V(x,\ y-h)) - 20\,V(x,\ y)\\ &+ V(x+h,\ y+h) + V(x-h,\ y+h) + V(x+h,\ y-h) + V(x-h,\ y-h)]\\ + N_0\,\frac{1}{6h^2}\,[&4\,(\phi(x+h,\ y) + \phi(x-h,\ y) + \phi(x,\ y+h) + \phi(x,\ y-h))\\ &- 20\,\phi(x,\ y)\\ &+ \phi(x+h,\ y+h) + \phi(x-h,\ y+h) + \phi(x+h,\ y-h) + \phi(x-h,\ y-h)]\\ = \Delta_\delta V(x,\ y) + N_0\,\Delta_\delta\phi(x,\ y) &= \Delta_\delta V(x,\ y) + N_0 \geq 0.\end{aligned}$$

**Verification of (*b*):** It is analogous to item (*a*).

Applying Theorem 4.1, item (*i*), to $V_+$ and $V_-$, we deduce that

$$V_+(x,\ y) \le V_+(x,\ y)\ \le V_+(x,\ y)$$

$$= [V(x,\ y)\ + N_0\ \phi(x,\ y)]$$

$$\le V(x,\ y)\ + N_0\ \frac{a^2+b^2}{2} \qquad (4.5)$$

and

$$V_-(x,\ y) \le [-\ V(x,\ y)]\ + N_0\ \frac{a^2+b^2}{2}, \qquad (4.6)$$

for all $(x,\ y) \in \Omega_\delta$.

From the definitions of $V_+$ and $V_-$, and the fact that $N_0$ and $\phi(x,\ y)$ are nonnegative, we obtain

$$V(x,\ y) \le V_+(x,\ y) \text{ and } -\ V(x,\ y) \le V_-(x,\ y),$$

for all $(x,\ y) \in \Omega_\delta \cup \partial\Omega_\delta$.

Combining this with (4.5), (4.6), and the definition of $N_0$, we conclude that

$$V(x,\ y) \le |V(x,\ y)|\ + \frac{a^2+b^2}{2}\left|\Delta_\delta V(x,\ y)\right|$$

and

$$-\ V(x,\ y) \le |V(x,\ y)|\ + \frac{a^2+b^2}{2}\left|\Delta_\delta V(x,\ y)\right|,$$

for all $(x,\ y) \in \Omega_\delta$. Consequently,

$$|V(x,\ y)| \le |V(x,\ y)|\ + \frac{a^2+b^2}{2}\left|\Delta_\delta V(x,\ y)\right|,$$

for all $(x,\ y) \in \Omega_\delta$.

Therefore,

$$|V(x,\ y)|\ \le |V(x,\ y)|\ + \frac{a^2+b^2}{2}\left|\Delta_\delta V(x,\ y)\right|.$$

Which completes the proof.

Finally, applying Theorem 4.2 to the discrete function $E_{i,j}$ defined in (4.1), we conclude that

$$\left|E_{i,j}\right| \leq \left|E_{i,j}\right| + \frac{a^2+b^2}{2}\left|\Delta_\delta E_{i,j}\right|.$$

Since $U_{i,j} = g(x_i, y_j) = u_{i,j}$ on the boundary $\partial\Omega_\delta$, we conclude that $E_{i,j} = 0$ on $\partial\Omega_\delta$. This implies that

$$\left|E_{i,j}\right| \leq \frac{a^2+b^2}{2}\left|\Delta_\delta E_{i,j}\right|. \qquad (4.7)$$

From (2.2), (3.1) and (4.1), we deduce that

$$\Delta_\delta E_{i,j} = \Delta_\delta\left(u_{i,j} - U_{i,j}\right) = \Delta_\delta u_{i,j} - \Delta_\delta U_{i,j} = \Delta_\delta u_{i,j} - f_{i,j}\, e^{\frac{h^2\Delta f_{i,j}}{12 f_{i,j}}} = T_{i,j}.$$

Using this result, together with (3.2) and (4.7), we obtain

$$\left|E_{i,j}\right| \leq C\, h^4, \qquad (4.8)$$

for some positive constant $C$, independent of $h$.

Therefore, from (3.2) and (4.8), we conclude that the numerical method (2.2) is convergent.

## 5. CONCLUSION

In this work, the consistency, stability, and convergence of the EFDM applied to two-dimensional nonlinear elliptic equations under Dirichlet boundary conditions in a domain $\Omega = (a, b)\times(a, b)$ were formally analyzed. The use of the compact nine-point stencil allowed obtaining a high-accuracy approximation for the Laplacian operator, increasing the order of accuracy relative to conventional five-point schemes.

The main results obtained in this work are highlighted below:

- **High-order consistency:** The Taylor series expansion with truncation at the sixth order guaranteed the fourth-order consistency of the discretization.
- **Central Role of the Discrete Maximum Principle:** The formal establishment of the Discrete Maximum Principle (Theorem 4.1) for the nine-point discrete operator constituted the indispensable foundation for the stability analysis of the scheme. From this principle, an *a priori* estimate (Theorem 4.2) was derived, bounding the norm of the discrete solution as a function of the difference operator.
- **Guarantee of Stability and Convergence:** By applying the *a priori* estimate – grounded in the Discrete Maximum Principle – to the global error, the stability and convergence of the numerical method were ensured.

It is concluded that the high-order exponential method grounded in the Discrete Maximum Principle combines theoretical soundness and computational efficiency for solving the analyzed class of problems. As future work, the extension of this approach to three-dimensional problems is suggested, as well as conducting numerical experiments for computational validation of the theoretical convergence rates.

## REFERENCES


CUMINATO, J. A.; JUNIOR, M. M. *Discretização de equações diferenciais parciais*: *técnicas de diferenças finitas*. 1. ed. Rio de Janeiro: SBM, 2013.

EVANS, L. C. *Partial differential equations*, v. 19. American mathematical society, 2022.

GILBARG, D., TRUDINGER, N. S. *Elliptic partial differential equations of second order*. Vol. 224. No. 2. Berlin: springer, 1977.

PANDEY, P. K. *A higher accuracy exponential finite difference method for the numerical solution of second order elliptic partial differential equations*. Journal of Mathematical and Computational Science, v. 3, n. 5, p. 1325-1334, 2013.

STRIKWERDA, J. C. *Finite Difference Schemes and Partial Differential Equations*. 2. ed. Philadelphia: SIAM, 2004.

THOMAS, J. W. *Numerical Partial Differential Equations: Finite Difference Methods*. New York: Springer-Verlag, 1995.